(*b*) If the four circles which touch S become points and S becomes a straight line, we have the common theorem on four collinear points A, B, C, D, viz: $AB \cdot CD + BC \cdot AD + CA \cdot BD = 0$.

(vii) A, B, C, D, E, F, are six points in any order on a conic section; and $ABCDEFA$ is an inscribed hexagram. If AB and DE, BC and EF, CD and FA intersect in X, Y, Z, respectively, then XYZ is a straight line, the Pascal line of the hexagram.

By moving vertices into coincidence, and assuming that their joins become tangents to the conic, we obtain this theorem in forms adapted to the pentagram, the tetragram, and the triangle.

Thus, for example, the hexagram becomes a triangle if B moves to A, D to C, and F to E. The theorem then becomes: If a triangle be inscribed in a conic, the sides intersect the tangents at the opposite vertices collinearly. The line of collinearity is the Pascal line of the triangle. Or the theorem may be stated thus: If a triangle be inscribed in a conic, and another circumscribed at the vertices of the first, the two triangles are in perspective. The axis of perspective is the Pascal line of the inscribed triangle.

This list of examples may be increased indefinitely from both elementary and advanced geometry. It will be found everywhere that general principles may be discovered to underlie groups of theorems which in our ordinary teaching of the subject have had no relationship to one another.

A NEW PROOF OF THE LAW OF TANGENTS.[1]

By WM. F. CHENEY, Jr., Berkeley, Cal.

1. In the triangle ABC, with angles A, B, and C, and sides a, b, and c, assume b greater than c. Lay off AD along b, equal to c, and draw BD; then

$$\angle DBA = \tfrac{1}{2}(180° - A) = \tfrac{1}{2}(B + C);$$

[1] Other geometrical discussions of the law of tangents may be found in the following sources: W. E. Johnson, *Treatise on Trigonometry*, London, 1889, p. 96; R. Levett and C. Davison, *Elements of Plane Trigonometry*, London, 1892, pp. 170–171 (also in J. W. Mercer, *Trigonometry for Beginners*, Cambridge, 1906, pp. 259–260); E. Brand, *Journal de mathématiques élémentaires* (de Longchamps), 1895, pp. 153–154; E. M. Langley, *Journal de mathématiques élémentaires* (de Longchamps), 1896, pp. 3–4 (construction introducing Wallace's Line); E. W. Hobson, *A Treatise on Plane Trigonometry*, second edition, Cambridge, 1897, pp. 155–156; E. J. Wilczynski, *Plane Trigonometry and Applications*, Boston, 1914, pp. 105–106; and J. W. Young and F. M. Morgan, *Plane Trigonometry and Numerical Computation*, New York, 1919, pp. 47–48.

The law of tangents in the usual form (except for notation) was first given by Vieta in the seventeenth century; *Francisci Vietæ opera mathematica*, Lugduni Batavorum, 1646, p. 402: "Vt adgregatum crurum ad differentiam eorundem, ita prosinus dimidiæ summæ angulorum ad basin ad prosinum dimidiæ differentiæ."

Cf. A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Teil 1, 1900, p. 188; Teil 2, pp. 44–45; and other places referred to under heading "Tangentensatz" in indexes.—Editor.

and

$$\angle\ CBD = \tfrac{1}{2}(B - C).$$

2. Draw AP, the perpendicular bisector of BD, and produce it to meet BC at Q; then

$$AP = BP \cdot \tan \tfrac{1}{2}(B + C);$$

and

$$PQ = BP \cdot \tan \tfrac{1}{2}(B - C).$$

3. Draw PR parallel to BC, bisecting CD at R; then

$$AR = \tfrac{1}{2}(b + c);$$

and

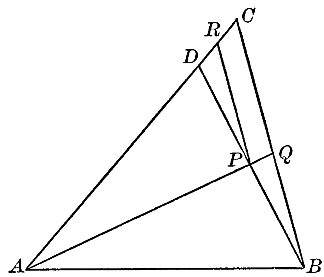
$$RC = \tfrac{1}{2}(b - c).$$

4. But

$$\frac{AR}{RC} = \frac{AP}{PQ};$$

therefore

$$\frac{b + c}{b - c} = \frac{\tan \frac{1}{2}(B + C)}{\tan \frac{1}{2}(B - C)}.$$



QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

Professor J. E. Trevor gave an example in the December number of the MONTHLY of an instance in which a theorem of analysis is of importance in thermodynamics. Another case of this sort is contributed by him as the first discussion in the present number. Corresponding to a change of independent variable, the form of a function is altered; it is readily shown that the rate of change can be represented as an ordinary derivative of another function closely related to the first. The notion is of use in connection with the heats of dilution of a solution. Instances in which mathematical conceptions arise in connection with applied science are always of value; even those mathematicians whose personal interests are principally in so-called "abstract" fields cannot afford to neglect the phases of mathematics sometimes termed "practical."

In the study of analytic geometry beyond the more elementary portions determinants play an extremely important part; a glance at any treatise on projective or metric analytic geometry, on differential geometry, or on non-Euclidean geometry analytically treated, will reveal page after page filled with resultants, discriminants, Hessians, linear dependence, and other ideas based on determinants. In the elementary parts of the subject, however, determinants